1. The thm in Lec 7:

K is decidable $\Leftrightarrow$ K and $\bar{K}$ are recognizable.

① A_{TM} is undecidable
② A_{TM} is recognizable
$\Rightarrow$ $\overline{A_{TM}}$ is unrecognizable.

which means. $\nexists M.\ L(M) = \overline{A_{TM}}$

$\Rightarrow L = \emptyset$.

$\emptyset$ is decidable.

2. By Rice's Thm:

L_P: $L(M) \in P$. where P = set of decidable languages.

① Semantic: ✓

② Non-trivial: ✓.

$\exists M_1$: $L(M)$ always rejects.

$\exists M_2$: $L(M) = A_{TM}$ is recognizable but undecidable.

$\Rightarrow$ Undecidable.

3.

$\exists M_1,\ L(M_1) \in P$
$\exists M_2,\ L(M_2) \notin P$

Assume L_P is decidable.
$\Rightarrow \exists D(\langle N \rangle) = \begin{cases} Acc, & L(N) \in P \\ Rej, & L(N) \notin P \end{cases}$

Core idea:

Construct N that (i = 1 or 2)

$L(N) = \begin{cases} L(M_i) & \text{when } M \text{ acc } w \\ \emptyset & \text{otherwise.} \end{cases}$

and $\mathbb{1}[\emptyset \in P] \neq \mathbb{1}[L(M_i) \in P]$

Then, use $D(N)$ to decide whether $L(N) \in P$
$\Leftrightarrow$ whether M acc w

Finally obtain a decider of A_{TM}. Contradiction.

$H(M, w)$:
  $N(x)$:
    Simulate M on w
    If M acc w:
      Simulate M_i on x: Return M_i's decision
    Else if M rej w: Rej // If M loops. N also loops.
  Return $D(\langle N \rangle)$

3. Semantic Non-trivial Decidable.

(a) $P = \{$ recognizable languages $\}$

✓ ✗ ✓ Just ac.

(b) ✗ ✓ ✓ Just check.

(c) It's not about $L(M)$.

M_1: Rejects ϵ $L(M_1) = L(M_2)$

M_2: Loops on ϵ.

Suppose L is recognizable.

$\exists\ U(\langle N \rangle) = \begin{cases} \text{Rej.} & N \text{ halts on } \epsilon \\ \text{Ac.} & \text{otherwise} \end{cases}$

$D(\langle M, w \rangle)$:

$N(x)$:

If M ac w:

Ac

If M rej w: Rej.

If M loops: Loops.

Return $U(\langle N \rangle)$

N will $\begin{cases} \text{Ac on } \epsilon. & M \text{ ac } w \\ \text{halts on } \epsilon & \text{otherwise} \end{cases}$

$U(\langle N \rangle) = \begin{cases} \text{Ac.} & M \text{ ac } w \\ \text{Rej.} & \text{otherwise.} \end{cases}$

Contradiction.

Undecidable

(d) $P = \{$ recognizable by some M of even states $\}$

✓ ✗ ✓ Just ac

(e) $P = \{$ recognizable by some M of ... $\}$

✓ ✓ ✗ Undecidable

4. ✓ ✓ ✗

5. Consider construct N that first simulates M, then loop until a perfect-number of steps to accept 01. $\Rightarrow$ Undecidable.

6. ✓ ✓ ✗

7. Consider construct N that first simulates M on it's first tape. then write a symbol on it's seconde tape if M ac w (if doesn't ac, no symbol on the second tape)
$\Rightarrow$ Undecidable.